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2005 J. Phys. A: Math. Gen. 38 1473

(http://iopscience.iop.org/0305-4470/38/7/005)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 1473–1490 doi:10.1088/0305-4470/38/7/005

Pure state correlations: chords in phase space

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Received 17 October 2004, in final form 18 October 2004
Published 2 February 2005
Online at stacks.iop.org/JPhysA/38/1473

Abstract

The intensity of the overlap of a quantum state with all its phase space translations defines its quantum correlations. In the case of pure states, these are invariant with respect to Fourier transformation. The overlaps themselves are here studied in terms of the Wigner function and its Fourier transform, i.e., the characteristic function or chord function. Unlike the Wigner function, the chord function need not be real, but eventual symmetry with respect to reflections about a phase space point may relate these representations. Semiclassical approximations for the 'classical-like' region of small chords and for large chords are derived. These lead to an interpretation of the Fourier invariance in terms of conjugate chords. The interrelation of large and small (sub-Planck) phase space structures previously noted in the literature is thus reinterpreted.

PACS numbers: 03.65.−w, 03.65.Sq, 03.65.Yz, 05.45.Mt

1. Introduction

Phase space correlations in quantum states have quite different properties from correlations in classical distribution functions. At the quantum level, the uncertainty principle imposes, on one hand, limitations on the possible distribution functions and at the same time creates strong relations between small and large scales that result in peculiar properties. Thus, for example, it has been realized [1] that the phase space area A over which a pure state extends determines the minimal size of the high frequency oscillation structures δA in the Wigner function by a kind of complementarity relationship,

$$A\delta A \gtrsim (2\pi\hbar)^2 \tag{1}$$

(for one degree of freedom), thus characterizing these structures as 'sub-Planck'. This relationship between possibly macroscopic areas A and sub-Planck areas δA is entirely due to the finiteness of Planck's quantum of action and is not present when considering the correlations of classical distribution functions. In general, and especially for systems with more than one degree of freedom, we shall show that it is more revealing to relate a given large displacement to a specific small scale oscillation, rather than to relate areas.

This complementarity between small and large scales in quantum distribution functions can be accessed by the Wigner function and its Fourier transform. Moreover, the latter, sometimes referred to as the characteristic function, the generating function, or simply the chord function, also lies at the core of a full representation of quantum mechanics, on a par with the Weyl–Wigner representation itself. Further considerations about the corresponding conjugate classical phase spaces as well as the notation are found in appendix A. In this paper we compare the way that both these representations exhibit quantum coherences on all scales, that are overlaid on purely classical structures. The copious previous literature on the Wigner functions allows us to concentrate on the properties of the chord representation.

In section 2 we present the intrinsic definition of phase space correlations for a general density operator $\hat{\rho}$ to be studied. These are then related to the chord function and the Wigner function in section 3. Then in section 4, we exploit the general interrelation of these functions in the case of pure states and present some simple examples. Section 5 is dedicated to the theory of small chords. This is the classical region, corresponding to a caustic of the Wigner function. The simple approximation thus obtained is rederived in the next section within the full semiclassical theory valid for large chords. Squaring the semiclassical chord function in section 7, we interpret the invariance of the correlations of a pure state with respect to the Fourier transform. This results from a geometrical conjugacy of chords defined on the classical phase space structures. Thus the complementarity of large and small chords arises as a limit among finite conjugate chords. The discussion in the last section recapitulates the full picture for extended pure states including 'ergodic states' of chaotic Hamiltonians.

2. Phase space correlations

The correlation between two states of a quantum system, represented by their density operators, $\hat{\rho}_A$ and $\hat{\rho}_B$, can be given an intrinsic definition as

$$C_{AB} \equiv \frac{\mathrm{tr}\,\hat{\rho}_A\hat{\rho}_B}{\sqrt{\mathrm{tr}\,\hat{\rho}_A^2\,\mathrm{tr}\,\hat{\rho}_B^2}}. \tag{2}$$

By the Schwartz inequality this quantity is always less than unity and for pure states, $\hat{\rho} = |\psi\rangle\langle\psi|$, it reduces to $C_{AB} = |\langle\psi_A|\psi_B\rangle|^2$. When the pair of states is obtained by the unitary evolution of slightly different Hamiltonians from the same initial state, the behaviour in time of C_{AB} reflects the regular or chaotic nature of the underlying classical motion [2–4]. If the states are related by a unitary transformation generated by an Hermitian operator $\hat{K}$,

$$\hat{\rho}_\alpha = \mathrm{e}^{-\mathrm{i}\alpha\hat{K}/\hbar}\,\hat{\rho}\,\mathrm{e}^{\mathrm{i}\alpha\hat{K}/\hbar}, \tag{3}$$

with α a real parameter, the small-α behaviour of the correlation is easy to ascertain:

$$C(\alpha) \equiv \frac{\mathrm{tr}\,\hat{\rho}\,\mathrm{e}^{-\mathrm{i}\alpha\hat{K}/\hbar}\hat{\rho}\,\mathrm{e}^{\mathrm{i}\alpha\hat{K}/\hbar}}{\mathrm{tr}\,\hat{\rho}^2} \approx 1 - \frac{\alpha^2}{\hbar^2}\frac{\mathrm{tr}[\hat{\rho}^2\hat{K}^2 - \hat{\rho}\hat{K}\hat{\rho}\hat{K}]}{\mathrm{tr}\,\hat{\rho}^2} = 1 + \frac{\alpha^2}{2\hbar^2}\frac{\mathrm{tr}[\hat{\rho},\hat{K}]^2}{\mathrm{tr}\,\hat{\rho}^2} \leqslant 1. \tag{4}$$

If $\hat{\rho}$ represents a pure state, this quadratic behaviour relates $C(\alpha)$ to the dispersion of the generator $\hat{K}$, i.e., $\langle\hat{K}^2\rangle - \langle\hat{K}\rangle^2$ [5].

In what follows we will be concerned with the phase space correlations of a quantum state produced by unitary translations in the L-dimensional phase space

$(p, q) = (p_1, \ldots, p_L, q_1, \ldots, q_L)$. Denoting the corresponding quantum operators by $(\hat{p}, \hat{q})$, the translation operators read

$$\hat{T}_\xi = \exp\left[\frac{\mathrm{i}}{\hbar}(\xi_p \cdot \hat{q} - \xi_q \cdot \hat{p})\right], \tag{5}$$

where the vector, or chord, $\xi = (\xi_p, \xi_q)$ represents an arbitrary direction in phase space. The expression

$$C_\xi \equiv \frac{\operatorname{tr} \hat{\rho} T_\xi \hat{\rho} T_\xi^\dagger}{\operatorname{tr} \hat{\rho}^2} = \frac{\operatorname{tr} \hat{\rho} \hat{\rho}_\xi}{\operatorname{tr} \hat{\rho}^2} \tag{6}$$

is an intrinsic definition of phase space correlation, quite independent of the representation used to compute it, and should not be confused with, e.g., *local* wavefunction correlations [6], which do depend on the specific coordinate representation.

In quantum optics it is customary to switch to the basis of creation and annihilation operators $(\hat{q} \pm \mathrm{i}\hat{p})/\sqrt{2\hbar}$. In this context, the translation operator (5) depends on the complex chords $(\xi_p \pm \mathrm{i}\xi_q)/\sqrt{2\hbar}$ and is called the displacement operator [7, 8]. The semiclassical limit for a complex phase space is not as transparent as the real theory treated here. However it is quite feasible to effect phase space translations in the optical context [9].

Some of the general formulae in the following sections are related to the theory of Gaussian noise channels. Indeed, the 'fidelity' of a Gaussian channel is just the correlation (6) averaged with a Gaussian window for the chords ξ [10].

3. The chord function and the Wigner function

The chord symbol for an operator $\hat{A}$ is defined as

$$A(\xi) \equiv \operatorname{tr} \hat{T}_{-\xi} \hat{A}, \tag{7}$$

allowing for the complete representation of $\hat{A}$ in terms of the unitary translations in phase space:

$$\hat{A} = \int \frac{\mathrm{d}\xi}{(2\pi\hbar)^L} A(\xi) \hat{T}_\xi. \tag{8}$$

In the case of the density operator $\hat{\rho}$, it is convenient to alter the normalization, so that

$$\chi(\xi) \equiv \frac{1}{(2\pi\hbar)^L} \operatorname{tr} \hat{T}_{-\xi} \hat{\rho} \tag{9}$$

is the definition of the chord function, also known as the characteristic function, or the generating function. (In classical mechanics the displacement ξ results from a trajectory of which it is the chord—see appendix A.)

The Fourier transform of the chord symbol is the Weyl symbol $A(X)$,

$$A(X) = \int \frac{\mathrm{d}\xi}{(2\pi\hbar)^L} \, \mathrm{e}^{-\mathrm{i}X \wedge \xi/\hbar} A(\xi), \tag{10}$$

in terms of the skew product

$$X \wedge \xi = P \cdot \xi_q - Q \cdot \xi_p. \tag{11}$$

In the case of the density operator, the Fourier transform of (9) is the familiar Wigner function. But, since the Fourier transform of the translation operator $\hat{T}_\xi$ itself corresponds to the classical reflection through the phase space point X [12],

$$\int \frac{\mathrm{d}\xi}{(2\pi\hbar)^L} \, \mathrm{e}^{\mathrm{i}X \wedge \xi/\hbar} \hat{T}_\xi = 2^L \hat{R}_X, \tag{12}$$

it follows that [11]

$$W(X) \equiv \frac{1}{(\pi\hbar)^L} \operatorname{tr} \hat{R}_X \hat{\rho}. \tag{13}$$

Though translations and reflections are quite distinct operators, they combine to form the affine group in geometry [13], which is transported into quantum mechanics by the operators $\hat{R}_X$ and $\hat{T}_\xi$ [12]. The family resemblance is striking when viewed, for instance, from the position representation:

$$2^L \hat{R}_X = \int \mathrm{d}\xi_q \left| Q + \frac{\xi_q}{2} \right\rangle \left\langle Q - \frac{\xi_q}{2} \right| \mathrm{e}^{\mathrm{i}P\cdot\xi_q/\hbar}, \tag{14}$$

whereas

$$\hat{T}_\xi = \int \mathrm{d}Q \left| Q + \frac{\xi_q}{2} \right\rangle \left\langle Q - \frac{\xi_q}{2} \right| \mathrm{e}^{\mathrm{i}\xi_p\cdot Q/\hbar}. \tag{15}$$

It is well known that the Wigner function cannot be identified with a probability distribution in phase space, even though this interpretation holds for marginal distributions and the calculation of averages of observables as phase space integrals. The main problem is that $W(X)$ can assume negative values and indeed they are present for all pure states that are not Gaussian (coherent) states. For this reason, it is also improper to refer to $\chi(\xi)$ as a characteristic function, even though its derivatives do generate moments of $\hat{q}$, $\hat{p}$, and any polynomial function in phase space, taking proper care of the operator ordering; for instance,

$$\langle q^n \rangle = \operatorname{tr} \hat{q}^n \hat{\rho} = (\mathrm{i}\hbar)^n \frac{\partial^n}{\partial \xi_p^n} \chi(\xi) \bigg|_{\xi=0}. \tag{16}$$

For the case of the identity operator, we obtain the normalization condition:

$$1 = \operatorname{tr} \hat{\rho} = \int \mathrm{d}X\, W(X) = (2\pi\hbar)^L \chi(0). \tag{17}$$

In the case of operators representing observables with smooth, classical-like Wigner functions, e.g., polynomials in q and p, their corresponding Fourier transforms, i.e., the chord symbols, are sharply localized (improper functions) close to the origin. On the other hand, the chord function for a normalized state is a proper function, and extends away from the origin, so as to represent truly quantum correlations. The process of decoherence, destroying the purity of the initial state, generically washes away the exterior structure of the chord function and compacts the mixed state onto the classical origin of chords. This has been shown for linear Markovian systems [14], and will be the subject of further work. Here we will describe the large and small scale features of the chord function and their close intertwining.

Combining definition (13) of the Wigner function with the group properties of the translation and reflection operators [12] it is easy to see that the Wigner function corresponding to the translated state $\hat{\rho}_\eta = \hat{T}_\eta \hat{\rho} \hat{T}_{-\eta}$ is

$$W_\eta(X) = W(X - \eta), \tag{18}$$

whereas the corresponding chord function is just

$$\chi_\eta(\xi) = \mathrm{e}^{\mathrm{i}\eta\wedge\xi/\hbar} \chi(\xi). \tag{19}$$

Unlike the Wigner function, the chord function is not necessarily real, but it may be real for a particular choice of phase space origin. It is shown in appendix B that the necessary and sufficient condition for this is that there exists a symmetry centre X, such that $[\hat{\rho}, \hat{R}_X] = 0$, and it is chosen as the origin. Since R_X has eigenvalues ± 1, the pure or mixed state $\hat{\rho}$ must then lie in the Hilbert subspace of either even or odd parity. For these parity-symmetric states, $\hat{\rho}_\pm$, the Wigner function and the chord function are obtained from each other by a mere rescaling:

$$W_\pm(X) = \pm 2^L \chi_\pm(-2X). \tag{20}$$

For general unsymmetric states, the real part of the chord functions is still determined by the diagonal part of $\hat{\rho}$ with respect to parity, whereas the imaginary part depends on the off-diagonal part. However, it is the intensity of the chord function, $|\chi(\xi)|^2$, that turns out to be most useful.

The intrinsic definition of phase space correlations C_ξ in (6) is readily translated into the properties of Wigner functions and chord functions:

$$\mathrm{tr}\,\hat{\rho}\hat{T}_\xi\hat{\rho}\hat{T}_\xi^\dagger = (2\pi\hbar)^L\int \mathrm{d}X\, W(X)W(X-\xi) = (2\pi\hbar)^L\int \mathrm{d}\eta\, \mathrm{e}^{\mathrm{i}\eta\wedge\xi/\hbar}|\chi(\eta)|^2. \tag{21}$$

Thus the correlations of the Wigner function can be identified with C_ξ and $|\chi(\eta)|^2$ is just the power spectrum of $W(x)$.

4. Pure states

It is worthwhile to recollect some examples for which the chord function can be identified with the Wigner function, once the origin is translated to the symmetry centre. In all the following cases we consider states of a harmonic oscillator with one degree of freedom and unit mass.

(i) Coherent states, $|\eta\rangle$, are displacements of the ground state of the harmonic oscillator by $\hat{T}_\eta$. The Wigner function is just a Gaussian centred on η,

$$W_\eta(X) = \frac{1}{\pi\hbar}\exp\left[-\frac{\omega}{\hbar}(Q-\eta_q)^2 - \frac{1}{\hbar\omega}(P-\eta_p)^2\right] \xrightarrow{\omega=1} \frac{1}{\pi\hbar}\,\mathrm{e}^{-(X-\eta)^2/\hbar}, \tag{22}$$

whereas, using (18) and (19),

$$\chi_\eta(\xi) = \frac{1}{2\pi\hbar}\exp\left(\frac{\mathrm{i}\eta\wedge\xi}{\hbar}\right)\exp\left[-\frac{\omega}{\hbar}\left(\frac{\xi_q}{2}\right)^2 - \frac{1}{\hbar\omega}\left(\frac{\xi_p}{2}\right)^2\right] \xrightarrow{\omega=1} \frac{1}{2\pi\hbar}\,\mathrm{e}^{\mathrm{i}\eta\wedge\xi/\hbar}\,\mathrm{e}^{-\xi^2/4\hbar}. \tag{23}$$

So, any translation of the coherent state merely alters the phase of the Gaussian chord function that sits on the origin.

(ii) A superposition of a pair of coherent states, $|\eta\rangle \pm |-\eta\rangle$, is sometimes known as a 'Schrödinger cat state'. Its Wigner function is (here and below we set $\omega = 1$)

$$W_\pm(X) = \frac{1}{2\pi\hbar(1\pm\mathrm{e}^{-\eta^2/\hbar})}\left[\mathrm{e}^{-(X-\eta)^2/\hbar} + \mathrm{e}^{-(X+\eta)^2/\hbar} \pm 2\mathrm{e}^{-X^2/\hbar}\cos\frac{2}{\hbar}X\wedge\eta\right]. \tag{24}$$

It consists of two 'classical' Gaussians centred on $\pm\eta$ and an interference pattern with a Gaussian envelope centred on their midpoint. The frequency of this oscillation increases with the separation $|2\eta|$. For the chord function,

$$\chi_\pm(\xi) = \frac{1}{4\pi\hbar(1\pm\mathrm{e}^{-\eta^2/\hbar})}\left[\mathrm{e}^{-(\xi/2-\eta)^2/\hbar} + \mathrm{e}^{-(\xi/2+\eta)^2/\hbar} \pm 2\mathrm{e}^{-\xi^2/4\hbar}\cos\frac{1}{\hbar}\xi\wedge\eta\right], \tag{25}$$

this same configuration has to be reinterpreted. Now the internal correlations of the individual coherent states are both superimposed onto the neighbourhood of the origin, as in (i), while their cross-correlation generates new Gaussians centred on the separation vectors $\pm 2\eta$. Of course, the general case of coherent states $|\eta_1\rangle$ and $|\eta_2\rangle$ merely leads to Gaussians centred on $\pm(\eta_1-\eta_2)$ with addition of the phase factor $\exp[\mathrm{i}(\eta_1+\eta_2)\wedge\xi/2\hbar]$.

(iii) Fock states, $|n\rangle$, i.e., the excited states of the harmonic oscillator, also have reflection symmetry with respect to the origin. Thus, from the exact Wigner function, first derived by Grönewold [15],

$$W_n(X) = \frac{(-1)^n}{\pi\hbar}\, \mathrm{e}^{-X^2/\hbar} L_n\left(\frac{2X^2}{\hbar}\right), \tag{26}$$

where L_n is a Laguerre polynomial, and (20) we obtain the chord function

$$\chi_n(\xi) = \frac{\mathrm{e}^{-\xi^2/4\hbar}}{2\pi\hbar} L_n\left(\frac{\xi^2}{2\hbar}\right). \tag{27}$$

It is interesting to note that the symmetry centre, which produces the maximum amplitude of the Wigner function, is nowhere near the classical manifold with energy $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. However, this point lies in a region of narrow oscillations, so that it does not affect the average of smooth observables.

All the above examples are singled out by some point of reflection symmetry, which must always be chosen as the origin for the chord function to be real. The chord function always assumes its maximum value $1/(2\pi\hbar)^L$ at the origin, whatever the symmetry. For a pure state the proof is immediate because

$$\langle\psi|\psi_\xi\rangle = (2\pi\hbar)^L \chi(-\xi), \tag{28}$$

which cannot have modulus greater than one, whatever the symmetry. But even an average of overlaps cannot exceed one, so $\chi(0)$ is also the maximum for mixed states. The Wigner intensity $[W(X)]^2$ need not have such a prominent peak in general. However we shall see in section 6 that the large scale features of the semiclassical forms of the Wigner function and the chord function maintain a mutual correspondence, even in the absence of a reflection symmetry.

General invariance with respect to Fourier transformation does hold for the correlation in the case of pure states. Indeed, combining (28) with (21) we obtain

$$(2\pi\hbar)^{2L} C_\xi = |\chi(\xi)|^2 = \int \frac{\mathrm{d}\eta}{(2\pi\hbar)^L}\, \mathrm{e}^{\mathrm{i}\eta\wedge\xi/\hbar} |\chi(\eta)|^2. \tag{29}$$

This is a remarkable property of all pure states and is in no way restricted by special symmetry properties that relate certain Wigner functions to their respective chord functions. All the same, we can start by considering the example of the chord functions so far studied: (i) in the case of a single Gaussian (22) the invariance is obvious, because the square modulus is a Gaussian with the appropriate width for its Fourier transform to be of the same form. (ii) For a pair of coherent states, the chord function is a sum of Gaussians. Its square modulus is also Gaussian and we again return to the same function by Fourier transformation. (iii) For Fock states this is not so obvious, but we can also write $|\chi(\xi)|^2$ as

$$|\chi(\xi)|^2 = \sum_k a_k \frac{\partial^k}{\partial\lambda^k}\, \mathrm{e}^{-\lambda\xi^2/2\hbar}\bigg|_{\lambda=1}, \tag{30}$$

with coefficients a_k that do not depend on ξ. So, the invariance of the ground state Gaussian entails that of all Fock states.

The Fourier invariance condition (29) includes as a special case the more familiar tracing over the full pure state condition $\hat{\rho}^2 = \hat{\rho}$. It can be easily checked that setting $\xi = 0$ in (29) gives the chord representation of the identity $\mathrm{tr}\,\hat{\rho}^2 = \mathrm{tr}\,\rho$. It follows that the difference of both sides of (29) for each chord ξ is a measure of the purity that generalizes the linear entropy. However, the loss of the phase information contained in the chord function, but absent in

$|\chi(\xi)|$, implies that these are necessary conditions, whereas the full sufficient condition of purity is only $\hat{\rho}^2 = \hat{\rho}$, which is expressed in the chord representation as

$$\int \mathrm{d}\eta\, \chi(\eta)\chi(\xi - \eta)\, \mathrm{e}^{\mathrm{i}\xi\wedge\eta/2\hbar} = \int \mathrm{d}\eta\, \chi_{\xi/2}(\eta)\chi(\xi - \eta) = \chi(\xi), \tag{31}$$

with $\chi_{\xi/2}(\eta)$ defined by (19).

5. Small chords

We now consider the chord function and the correlation function of semiclassical states, i.e., those related to generalized Bohr–Sommerfeld energy levels. This section focuses on the limit of small chords.

Starting from the Wigner–Weyl representation we may rewrite

$$\chi(\xi) = \frac{1}{(2\pi\hbar)^L}\langle \hat{T}_{-\xi}\rangle = \frac{1}{(2\pi\hbar)^L}\int \mathrm{d}X\, T_{-\xi}(X)W(X). \tag{32}$$

Besides the Wigner function, we have here introduced the Weyl symbol for the translation operator:

$$2^L \mathrm{tr}\, \hat{T}_{-\xi}\hat{R}_X \equiv T_{-\xi}(X) = \mathrm{e}^{-\mathrm{i}\xi\wedge X/\hbar}. \tag{33}$$

If ξ is small enough, i.e., $|\xi| \lesssim \hbar$, then $T_\xi(X)$ behaves like a smooth, classical-like symbol. It still oscillates, but with a classical wavelength. In such a case, to a good approximation, we can replace the Wigner function in (32) by the simplest semiclassical expression [16]

$$W_{\mathcal{I}}(X) \simeq \frac{1}{(2\pi)^L}\delta(I(X) - \mathcal{I}). \tag{34}$$

Here $I(X)$ is the set of L action variables for an integrable system with L degrees of freedom and $\mathcal{I}$ is the set of quantized action values for this particular state [16, 19]. Within this approximation, the average of a quantum observable $\hat{A}$ is just a purely classical average over a torus:

$$\langle \hat{A}\rangle \simeq \int \mathrm{d}X\, A(X)\frac{\delta(I(X) - \mathcal{I})}{(2\pi)^L} = \int \frac{\mathrm{d}\theta}{(2\pi)^L}A(\theta), \tag{35}$$

where θ are the angle variables conjugate to the actions, which describe positions on the quantized torus, and $A(\theta) = A(X(\theta))$. In the case of the representation for the chord function (32), we obtain

$$\chi(\xi) \simeq \frac{1}{(2\pi\hbar)^L}\int \frac{\mathrm{d}\theta}{(2\pi)^L}\, \mathrm{e}^{-\mathrm{i}\xi\wedge X(\theta)/\hbar}. \tag{36}$$

Certainly, a bad choice of origin will lead to large phases in (36), but we have already studied the trivial phase change due to translating the origin. Thus, we can increase the quality of (36) by choosing the origin to minimize $|X(\theta)|$ on average.

Let us check the semiclassical approximation (36) for the simplest case of the Fock states, discussed in the previous section. Choosing $\omega = 1$, we have action-angle variables that are merely canonical polar coordinates

$$q = \sqrt{2\mathcal{I}}\cos\theta, \qquad p = \sqrt{2\mathcal{I}}\sin\theta. \tag{37}$$

Thus, choosing ξ along the p-axis, without loss of generality, we obtain

$$\chi(\xi) \simeq \frac{1}{2\pi\hbar}\int \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{e}^{-\mathrm{i}\sqrt{2\mathcal{I}}|\xi|\cos\theta/\hbar} = \frac{1}{2\pi\hbar}J_0\left(\frac{\sqrt{2\mathcal{I}}|\xi|}{\hbar}\right), \tag{38}$$

where J_0 is a Bessel function. This result should be a good approximation to the exact formula (27) for $|\xi| \lesssim \hbar$ and large $n = \mathcal{I}/\hbar - 1/2$. To check this, first note that $\chi(\xi)$ oscillates with a wavelength $\lambda \sim \hbar/\sqrt{\mathcal{I}}$. In terms of this scale, and defining $z = |\xi|/\lambda$, the argument of the Laguerre polynomial in (27) reads

$$\frac{\xi^2}{2\hbar} = \frac{z^2}{2n}. \tag{39}$$

Hence, for $z^2 \ll n$, we recover (38) by using the formula [17]

$$\lim_{n\to\infty} L_n\left(\frac{z^2}{2n}\right) = J_0(\sqrt{2}z). \tag{40}$$

In the next section we shall study the semiclassical limit of the chord function for large chords and show that in the particular case of the harmonic oscillator this coincides with the expansion of (38) for large argument:

$$J_0(y) \approx \frac{2}{\sqrt{\pi y}} \cos\left(y - \frac{\pi}{4}\right). \tag{41}$$

6. Semiclassical theory

The easiest path to obtain the semiclassical form of the chord function for finite chords is to start from the general WKB wavefunctions. The underlying classical structure is assumed to be a curve (for $L = 1$) or a Lagrangian surface ($L > 1$) defined by the actions I. Fixing the action, such structure can be described locally as a function $p = p(q, I))$ which may have several branches. Thus we define

$$S(q, I) = \int_{q_0}^{q} p(q', I)\,\mathrm{d}q', \tag{42}$$

which is the generating function for the transformation $(p, q) \to (I, \theta)$, such that

$$\frac{\partial S}{\partial I} = \theta, \qquad \frac{\partial S}{\partial q} = p. \tag{43}$$

Then the corresponding WKB wavefunctions are linear combinations of

$$\langle q|\psi_I\rangle = c\left|\det \frac{\partial^2 S(q, I)}{\partial q \partial I}\right|^{1/2} \mathrm{e}^{\mathrm{i}S(q,I)/\hbar}, \tag{44}$$

for the various branches of $S(q, I)$, with c a normalization constant [18, 19].

The similarity between equations (14) and (15) for the operators that determine the chord function and the Wigner function, allows us to follow the same steps as Berry [16] for the semiclassical Wigner function. Thus we presume that

$$\chi(\xi) = \frac{c^2}{(2\pi\hbar)^L} \int \mathrm{d}Q' \exp\left\{\frac{1}{\hbar}\left[S\left(Q' + \frac{\xi_q}{2}, I\right) - S\left(Q' - \frac{\xi_q}{2}, I\right) - \xi_p Q'\right]\right\}$$
$$\times \left|\det \frac{\partial^2 S(Q' + \xi_q/2, I)}{\partial q \partial I} \det \frac{\partial^2 S(Q' - \xi_q/2, I)}{\partial q \partial I}\right|^{1/2} \tag{45}$$

has a stationary point within a single of branch of $S(q, I)$. The stationary phase condition for $Q' = Q$

$$p_{\mathcal{I}}\left(Q + \frac{\xi_q}{2}\right) - p_{\mathcal{I}}\left(Q - \frac{\xi_q}{2}\right) = \xi_p \tag{46}$$

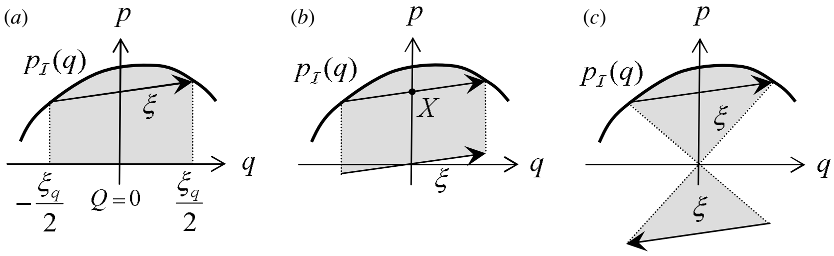

Figure 1. The stationary phase condition for the semiclassical chord function $\chi(\xi)$ is just that ξ be a geometrical chord for the curve $I(x) = \mathcal{I}$, locally given by $p_{\mathcal{I}}(q)$. The stationary phase itself is given by the shaded area in (*a*). The constructions in (*b*) and (*c*) are invariant with respect to linear transformations and have the same areas.

specifies that the geometrical chord corresponding to the arc of $p_{\mathcal{I}}(q) = p(q, I = \mathcal{I})$, lying between $q = Q - \xi_q/2$ and $q = Q + \xi_q/2$, coincides with the given chord ξ.

Unlike the Wigner function, the phase of the chord function depends on the choice of phase space origin, but this phase is trivially specified by the overall phase factor (19) for a translation by a vector η. Thus, let us for now assume that the origin is translated to $(0, Q)$. Then the stationary phase action is just the area shown in figure 1(*a*). The alternative constructions (*b*) and (*c*), also shown in figure 1, have the same area and enjoy the advantage that they survive arbitrary linear canonical transformations which preserve the origin. In other words, we may always consider the phase for the semiclassical chord functions to be defined by a sum of areas: (i) the area sandwiched between the chord and the arc into which if fits, $S(X, \mathcal{I})$. This is just the same as for the semiclassical Wigner function, except that there the construction starts from the centre of the chord X (figure 1(*b*)), rather than from the chord itself. (ii) The area that is added to this may be taken as that of the parallelogram obtained by transporting the centre of ξ to the origin (figure 1(*b*)), or the polygonal figure of eight constructed by the chord and its reflection around the origin (figure 1(*c*)). In either case the value of the area is just $X \wedge \xi$. Therefore we construct the chord generating function

$$S(\xi, I) = S(X, I) - X \wedge \xi, \tag{47}$$

which is just the Legendre transformation of the centre action, appropriate to the Wigner function, as discussed in appendix A. This geometry is immediately generalized for $L > 1$: the action for any polygonal figure is just the algebraic sum of the areas of its projections on each conjugate plane. Furthermore, it does not matter which arc is chosen between the tips of ξ along $p_{\mathcal{I}}(q)$ because this is a Lagrangian surface.

Thus, the contribution of each realization of the chord ξ, with centres X_j and closing arcs γ_j on the quantized curve or Lagrange manifold $I(x) = \mathcal{I}$, has the form

$$\chi_j(\xi) = A_j(\xi) \exp\left[\frac{\mathrm{i}}{\hbar} S_j(\xi) - \sigma_j \frac{\pi}{4}\right]. \tag{48}$$

Here σ_j is the signature of the matrix

$$\frac{\partial^2}{\partial Q^2} \left[S(q_+, I) - S(q_-, I)\right], \tag{49}$$

where $q_\pm = Q \pm \xi/2$, so that in the case of the curve in figure 1 we have $\sigma = +1$.

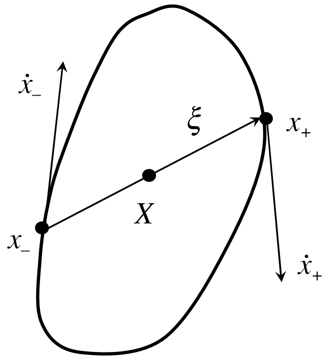

Figure 2. The semiclassical chord function, $\chi(\xi)$, is determined by the geometrical chord ξ, centred on X, and joining the points x_- and x_+ on the quantized classical curve. The amplitude depends on both phase space velocities at the tips, as explained in the text.

The amplitude for each realization of the chord, within a normalization factor, is given by

$$|A_j(\xi)|^2 = \left|\det\left[\frac{\partial p_{\mathcal{I}}}{\partial q}(q_+) - \frac{\partial p_{\mathcal{I}}}{\partial q}(q_-)\right]\det\frac{\partial I}{\partial p}(p(q_+), q_+)\det\frac{\partial I}{\partial p}(p(q_-), q_-)\right|$$
$$= \left|\det\left[\left.\frac{\partial I}{\partial p}\right|_+\left.\frac{\partial I}{\partial q}\right|_- - \left.\frac{\partial I}{\partial p}\right|_+\left.\frac{\partial I}{\partial q}\right|_-\right]\right|, \tag{50}$$

because

$$\frac{\mathrm{d}I}{\mathrm{d}q}(p(q), q) = \frac{\partial I}{\partial q} + \frac{\partial I}{\partial p}\frac{\partial p}{\partial q} = 0, \tag{51}$$

along the classical surface. These amplitudes coincide with those of the Wigner function evaluated at the point $X_j(\xi)$. In the case where $L = 1$, the 1×1 determinant is only a single factor. Taking $I(x)$ to be a Hamiltonian, the corresponding phase space velocity tangent to the phase space curve is

$$\dot{x}_I = J\frac{\partial I}{\partial x}, \tag{52}$$

where J is the symplectic matrix (A.4), and we may interpret the amplitude of the chord function as

$$|A_j(\xi)|^{-2} = (\dot{x}_I)_+ \wedge (\dot{x}_I)_- = \{I_+, I_-\}, \tag{53}$$

where the displaced actions are defined by

$$I_\pm(X) = I\left(X \pm \frac{\xi}{2}\right) \tag{54}$$

(see figure 2). In the general case each element of the determinant is such a Poisson bracket for the different L action variables [20].

For a maximal chord, termed a diameter, such that $\dot{x}_-$ becomes parallel to $\dot{x}_+$, the amplitude diverges to generate a caustic of the chord function. Beyond this boundary $\chi(\xi)$ becomes negligible. In the limit of small chords, $\dot{x}_\pm$ are almost parallel, and the amplitude may be arbitrarily large. In this region the above semiclassical theory ceases to operate and, because the chord normalization condition is precisely $\chi(0) = 1/2\pi\hbar$, we have neglected to define the overall normalization factor. Actually this limit around the origin is much nastier than normal semiclassical caustics, since the entire classical manifold can be defined by a succession of infinitesimal chords. It is thus a nontrivial problem to construct a semiclassical uniform theory including large and small chords.

However, it is possible to connect the semiclassical theory to that for the small chords of the previous section. Let us take the case that the given chord, ξ, is parallel to the q-axis, so that we can rewrite (45) as

$$\chi(\xi) = \frac{c^2}{(2\pi\hbar)^L} \int \mathrm{d}Q \exp\left\{\frac{\mathrm{i}}{\hbar}\left[S\left(Q + \frac{\xi_q}{2}, I\right) - S\left(Q - \frac{\xi_q}{2}, I\right)\right]\right\} \times \left|\det\frac{\partial\theta}{\partial Q}\right|_+^{1/2} \left|\det\frac{\partial\theta}{\partial Q}\right|_-^{1/2}, \tag{55}$$

using (43). Expanding

$$\theta(q_\pm) = \theta(Q) \pm \frac{\partial\theta}{\partial Q}\frac{\xi}{2} + \cdots, \tag{56}$$

we obtain

$$\left|\det\frac{\partial\theta}{\partial Q}\right|_-^{1/2} \left|\det\frac{\partial\theta}{\partial Q}\right|_+^{1/2} = \left|\det\frac{\partial\theta}{\partial Q}\right| \left[1 + \mathcal{O}(\xi^2)\right] \tag{57}$$

whereas

$$S\left(Q + \frac{\xi_q}{2}, I\right) - S\left(Q - \frac{\xi_q}{2}, I\right) = p_I(Q)\xi_q + \mathcal{O}\left(\xi_q^3\right). \tag{58}$$

Hence, we can cancel the amplitudes in (55) by changing the integration variable, $Q \to \theta$:

$$\chi(\xi) \simeq \frac{1}{(2\pi\hbar)^L} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{e}^{\mathrm{i}X(\theta)\wedge\xi/\hbar}. \tag{59}$$

The general form of (59) holds for any choice of origin and direction of the small chord ξ.

We have here rederived the small chord expression for the chord function used in the last section in a way that has several advantages. First we note that Berry's original derivation of the δ-function approximation for the Wigner function also linearized the action for small chords [16]. But here the chord is indeed fixed and can be assumed to be small, whereas there the integral is over all chords. We should note that (34), or the extension of (59) for all chords, is tantamount to approximating

$$\hat{\rho} \approx 2^L \int_0^{2\pi} \frac{\mathrm{d}\theta}{(2\pi)^L} \hat{R}_{X(\theta)}. \tag{60}$$

We can now evaluate the error in (59) when the chord ξ is large enough for both this integral and (45) to be evaluated by stationary phase. Consider the convex curve $I(x) = \mathcal{I}$ in figure 3(a). The difference in the phases for the stationary phase evaluation at each stationary point of (59) or the full integral (45) is the part of the area near the corner of the circumscribed parallelogram with two sides ξ tangent to the curve, lying outside the curve. If we approximate the curve as a parabola around the point of tangency $\delta p = a(\delta q)^2$, the leftover area is just $A(\xi) = 2a|\xi|^3/3$. So, if the diameter between the parallel tangents with the direction ξ is $\zeta(\xi)$, the true stationary area for each realization of the chord is just

$$S(\xi) = \frac{1}{2}\zeta \wedge \xi - \frac{2a|\xi|^3}{3}. \tag{61}$$

Thus the stationary phase evaluation of the full integral is valid as long as $|\zeta \wedge \xi| > \hbar$, whereas the integral (59) can be used if $a|\xi|^3 < \hbar$. Therefore the range of overlap for both

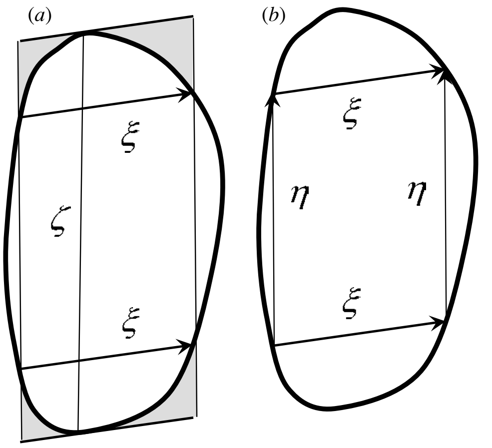

Figure 3. (*a*) The difference in the pair of stationary phases for (59) is proportional to the parallelogram that has ξ as one of its sides and is doubly tangent to the quantized curve. The difference between this area and the pair of stationary areas in the full semiclassical formula is just the shaded area at the corners. (*b*) A pair of stationary phase realizations in a convex quantized curve for a finite chord ξ defines an inscribed parallelogram. The other side is the conjugate chord η. The sign of the conjugate chords plays no role in the equations for $|\chi(\xi)|^2$ because $\chi(\xi)^* = \chi(-\xi)$.

approximations is

$$\frac{\hbar}{|\zeta|} \lesssim |\xi| \lesssim \left(\frac{\hbar}{a}\right)^{1/3}, \tag{62}$$

so that we can always make the transition between both approximations as $\hbar \to 0$.

7. Conjugate chords

The pair of parallel tangents shown in figure 3(*a*) can be defined as the realizations of an infinitesimal chord for the curve $I(x) = \mathcal{I}$. The points where the tangents touch this curve are joined by the diameter ζ. In the case of a finite chord ξ, we can also define the conjugate chord, η, as that which closes off the inscribed parallelogram with the sides ξ, as shown in figure 3(*b*). Clearly, the area of this parallelogram is $\xi \wedge \eta$, which is smaller than $\xi \wedge \zeta$, used in the previous section. Choosing the origin at the centre of this parallelogram, we obtain the phase of each contribution to $\chi(\xi)$ as $\xi \wedge \eta/2$, added to each of the chord areas for ξ.

Increasing the length of ξ while keeping its direction constant, the pair of realizations eventually coalesce along a diameter of the closed curve. Thus we obtain in this limit an interchange in the role of the 'conjugate chords' ξ and η. Now it is the small chord η that determines the phase of the contribution as the chord ξ approaches the caustic of $\chi(\xi)$ at its diameter.

It is not only in the geometry of the curve that the conjugate chords ξ and η interact. Indeed, the phase difference between both realizations of the chord ξ coincides with the sum of the chord areas for the realizations of the chord η. To see this, note that this phase difference is just the shaded area in figure 4(*a*) divided by Planck's constant. But, because this curve is quantized (its enclosed area is an integer factor of $2\pi\hbar$, plus a Maslov correction), the phase difference for ξ is the same as for the unshaded area. Note now that the phase difference can also be obtained by translating the whole quantized curve by the chord ξ and measuring the

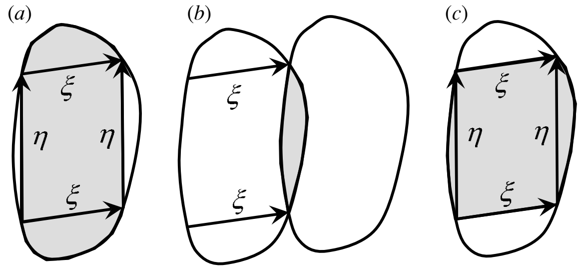

Figure 4. The phase difference for the pair of realizations of the chord ξ is the shaded area in (a) divided by $\hbar$. The quantization of the curve equates this phase (modulo 2π) to the phase for the shaded area in (b), obtained by the overlap of the curve with its translation by ξ. Adding to this the area of the inscribed parallelogram, we obtain the phase difference for the pair of realizations of the chord η in (c).

overlap. This is just the classical correspondence for the definition of $\chi(\xi)$ of a pure state in (28). If the classical quantized curves all have a symmetry centre, it makes no difference whether or not the curve is reflected about its centre prior to translation. In this way we obtain a semiclassical version of the trivial relation between the Wigner function and the chord function in the case of centre-symmetrical systems. The overlap between the quantized curves, or quantized tori, was already employed to study the semiclassical approximation to the Wigner function [20].

Both the Wigner functions and the chord function can be considered as special cases of the overlap of two semiclassical states $|\psi\rangle$ and $|\phi\rangle$, corresponding to classical curves, or Lagrangian surfaces. In the case of the Wigner function $|\phi\rangle = \hat{R}_X|\psi\rangle$, whereas, for the chord function $|\phi\rangle = \hat{T}_\xi|\psi\rangle$. In the general formalism of Littlejohn [21], each intersection of the manifolds corresponding to $|\psi\rangle$ and $|\phi\rangle$ determines a semiclassical contribution. The phase difference between these contributions is determinated by the overlap area, just as in figure 4(b), within a Maslov correction. The amplitude of each semiclassical term is then given by Poisson brackets between both manifolds at the intersections. Obviously, in the present case, this coincides with the Poisson brackets for the same manifold at either tip of the chord, as obtained in (53). Perhaps this general point of view provides the more pleasing explanation for the identical form of the Wigner and the chord amplitudes.

The relation between conjugate chords, ξ and η, and the corresponding evaluations of the semiclassical chord function for the same state $|\psi\rangle$, i.e., $\chi_\psi(\xi)$ and $\chi_\psi(\eta)$, is mediated by the parallelogram formed by ξ and η. The phase difference $\Delta S(\xi)$, in units of $\hbar$, between both contributions to $\chi_\psi(\xi)$ can be pictured as either the shaded area in figure 4(a), or the unshaded area corresponding to both realizations of the chord η. But, if we now add $\xi \wedge \eta$ to these unshaded areas, we obtain the new shaded area $\Delta S(\eta)$ in figure 4(c), which determines the phase difference of both contributions to $\chi_\psi(\eta)$.

Squaring the semiclassical approximation for $\chi(\eta)$ we obtain

$$(2\pi\hbar)^2 C_\eta = |\chi(\eta)|^2 \simeq \{I_+, I_-\}_1^{-1} + \{I_+, I_-\}_2^{-1} + \{I_+, I_-\}_1^{-\frac{1}{2}}\{I_+, I_-\}_2^{-\frac{1}{2}} \cos\frac{\Delta S(\eta)}{\hbar} \tag{63}$$

in the simple case where $I(x) = \mathcal{I}$ is a convex curve, so that there is a pair of realizations for the chord, η, with intensities given by (53). Inserting (63) into the Fourier integral (29), we find that the first two smooth terms only contribute to the classical neighbourhood of the origin. It turns out that the stationary condition for η in (29) for a given value of ξ in the last term is precisely that η be the conjugate chord to ξ, i.e., that ξ and η form an inscribed parallelogram in the curve corresponding to $|\psi\rangle$. This is a consequence of equation (46).

Therefore the Fourier invariance of the quantum correlation results semiclassically from the relation of the chord function $\chi(\xi) \leftrightarrow \chi(\eta)$, for conjugate chords.

In the limit as the family of chords with a fixed direction approaches its maximum value for a given quantized curve, i.e., a diameter, the conjugate chord approaches the origin in the direction of the parallel tangents at the tips of the diameter. However, in this region, the simple semiclassical amplitudes A_j given by (53) become singular and each of the three terms in (63) contributes. It is here necessary to replace the semiclassical contribution by uniform approximations in terms of Airy functions. This will be the subject of further work.

The role of diameters as conjugate chords near the origin is brought forth by combining (29) with (36), for $|\xi| \to 0$:

$$C_\xi \approx \int \frac{\mathrm{d}\theta_+}{(2\pi)^L} \frac{\mathrm{d}\theta_-}{(2\pi)^L} \, \mathrm{e}^{\mathrm{i}\xi\wedge[x(\theta_+)-x(\theta_-)]/\hbar}. \tag{64}$$

We can interpret $x(\theta_+) - x(\theta_-)$ as the set of chords supported by the quantized curve, or L-dimensional torus. This confirms the role of the first two terms of (63) which contribute to the Fourier transform of C_ξ for small ξ: all the chords supported by the curve contribute. However, as soon as ξ grows in modulus enough that we may evaluate (64) by stationary phase (condition (62)), the dominant contribution comes from the diameter $\zeta(\xi)$, i.e., $\zeta = x(\theta_+) - x(\theta_-)$, such that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta_+} = \frac{\mathrm{d}x}{\mathrm{d}\theta_-} = 0; \tag{65}$$

in other words, the tangents at $\theta_\pm$ are parallel.

Up to now we have analysed cases where the classical structure is a continuous curve or surface in phase space. Let us now consider an alternative classical setting, a state

$$|\psi\rangle = \sum_j a_j |\eta_j\rangle, \tag{66}$$

where $|\eta_j\rangle$ are coherent states centred on the phase space points η_j. Then the density operator is

$$|\psi\rangle\langle\psi| = \sum_j |a_j|^2 |\eta_j\rangle\langle\eta_j| + \sum_{j\neq k} a_j a_k^* |\eta_j\rangle\langle\eta_k|, \tag{67}$$

and its chord representation $\chi_\psi(\xi)$ is a simple generalization of (24): each chord $\pm(\eta_j - \eta_k)$ is the centre of a Gaussian, whereas the diagonal terms in (67) interfere collectively in the neighbourhood of the origin. Taking $|\chi_\psi(\xi)|^2$, we again have Gaussians centred at $\eta_j - \eta_k$ and at the origin. It is easy to see that the Fourier transform merely interchanges the contributions to the origin with the pair of Gaussians centred at $\pm(\eta_j - \eta_k)$. However, there is a new contribution to the Fourier transform if the four vectors $\eta_1, \eta_2, \eta_3, \eta_4$ form a parallelogram, i.e., $\eta_2 - \eta_1 \simeq \eta_3 - \eta_4$ and consequently $\eta_3 - \eta_2 \simeq \eta_4 - \eta_1$, where $\simeq$ means that the vectors differ by $\mathcal{O}(\sqrt{\hbar})$. Then the Fourier transform of $|\chi(\xi)|^2$ for $\xi = \eta_1 - \eta_4$ receives contributions from the neighbourhood of $\eta_2 - \eta_1$ and vice versa. Thus, again we verify the role of conjugate chords in the Fourier invariance of the quantum correlations. The consistency of fitting semiclassical states with Gaussian coherent states has been recently investigated in [22].

8. Discussion: resurgence of pure state correlations

We recapitulate our findings: as we displace a pure state that is classically extended, i.e. that quantizes with a large quantum number, the correlation in a given direction goes through four different stages: (a) an initial, relatively simple stage of very short chords in which the behaviour is quadratic in the displacement and determined purely by the phase space extent of the state. This is the region that has received most attention in the literature so far [1, 5, 23]. (b) A second oscillatory stage ruled by the points of intersection of the two displaced tori, with relative phases that are semiclassically determined. (c) A third stage characterized by a chord caustic where the semiclassical contributions diverge and where uniform approximations are still needed. (d) An asymptotic region where the correlation decays uniformly to zero. We have here given special attention to the transitions between stages (a) and (b). Other large systems such as a widespread superposition of coherent states will also have large classical values for $\langle p^2\rangle$ or $\langle q^2\rangle$. For all such pure states the initial decay of the correlations with growing displacement will be followed by their oscillatory resurgence for those chords that cause the underlying classical structures to overlap.

It is important to consider higher phase space dimensions, since all our examples were restricted to a single degree of freedom. Again, it is clear that in the case of arbitrary superpositions of pairs of coherent states the straight Fourier analysis of the Wigner function leads to very similar pictures for the pair of classical-like Gaussians in the Wigner function and the peaks of correlations at the chords that separate them.

Integrable systems of higher dimension might appear to be harder to analyse, but this is not so. It was shown in [20] that the Wigner function corresponding to L-dimensional quantized tori is characterized by a Wigner caustic of dimension $2L - 1$ in which the torus appears as a higher singularity. At this boundary between an oscillatory inner region of phase space and the evanescent region outside, the Wigner function attains an amplitude maximum, which can be described locally by an Airy function and its derivative. In the case that there exists a reflection symmetry centre, the chord function must be identical to the Wigner function within a phase and the rescaling (20). Even without such a reflection symmetry, the region where a finite L-torus and its rigid translation intersect is bounded by a $(2L - 1)$-dimensional surface at which they touch nontransversally—the chord caustic.

Finally we must consider the case of an eigenstate of a chaotic Hamiltonian. In this case Shnirelman's theorem [24] guarantees that most states are ergodic in the sense that the average of smooth functions of the observables $\hat{p}$ and $\hat{q}$ is given by a classical average over the energy shell. In its simplest form, quantum ergodicity may be taken as the Berry–Voros hypothesis that the Wigner function is approximately a Dirac delta function on the energy shell [6, 25]. It follows immediately that for small chords we may adapt the discussion in section 5 to obtain the chord function for the nth energy eigenstate as

$$\chi_n(\xi) = \langle \mathrm{e}^{\mathrm{i}\xi\wedge\hat{x}}\rangle_n = \frac{1}{(2\pi\hbar)^L}\frac{\int \mathrm{d}x\,\delta(H(x)-E_n)\,\mathrm{e}^{\mathrm{i}\xi\wedge x/\hbar}}{\int \mathrm{d}x\,\delta(H(x)-E_n)} \tag{68}$$

(see also [5, 23]). Conjugate to this short chord behaviour there must be appreciable large arguments for which $\chi_n(\xi)$ is also large, corresponding to the largest chord fitting between a pair of points in the energy shell for any given phase space direction. Once again, in the case of a reflection symmetric chaotic Hamiltonian, we may invoke the (scaled) identity with the Wigner function which is dominated by the energy shell itself, according to Shnirelman's theorem [24].

In all the extensions to higher dimensions, large chords ξ generate oscillations in the Wigner function with wave vector $J\xi$. In the case of a semiclassical state constructed on a Lagrangian torus, the fine structure in the Wigner function results from the interference of a finite number of locally plane waves. The fine structure of ergodic chaotic Wigner functions has not yet been explored.

It should be noted that our general description is in no way limited to stationary states. The curves and surfaces which we have mainly treated may be evolving classically, while the corresponding quantum system also evolves. We have not treated this dynamics here, but our description is valid for any snapshot. Even for the evolution of a localized wave packet in a chaotic system, it is possible to advance that long scale quantum correlations develop as the classical packet spreads over the energy shell.

This study of the coherence properties of pure states suggests that it is not always profitable to transform back to the centre phase space of the Wigner function from the phase space of chords, since it is here that the full structure of chord conjugacies is manifest. Furthermore, there is a decided advantage to allowing the collapse of all classical information onto the neighbourhood of $\xi = 0$. All that extends out in the chord phase space are signs of quantum coherence and these are structures that are lost in the nonunitary evolution that results from tracing out the interaction with an uncontrollable environment.

Acknowledgments

We thank O Brodier for interesting comments. Partial financial support from Millenium Institute of Quantum Information, PROSUL, CAPG-BA (CAPES) and CNPq is gratefully acknowledged.

Appendix A. Conjugate phase spaces

The points in phase space for a system with L degrees of freedom are here denoted simply by $x = (p, q) = (p_1, \ldots, p_L, q_1, \ldots, q_L)$. In the classical limit, $\hbar \to 0$, we may associate a quantum state with such a phase point, but we need pairs of points (x_-, x_+) to describe operators $\hat{A}$, such as $\langle q_+|\hat{A}|q_-\rangle$, or $\langle p_+|\hat{A}|p_-\rangle$. In both these cases, we only use, in fact, half of the phase space variables, as consistent with the uncertainty principle.

Another alternative is to use either the *centre*

$$X = (P, Q) = (x_+ + x_-)/2 \tag{A.1}$$

or the *chord*

$$\xi = \left(\xi_p, \xi_q\right) = x_+ - x_-. \tag{A.2}$$

As shown in [12, 16] we may then identify the argument in the Weyl symbol of $\hat{A}$, $A(X)$, with the centre (A.1). In the same way, the argument in the chord symbol $A(\xi)$ corresponds to the chord (A.2). In the case of a unitary transformation, corresponding classically to a trajectory, we identify ξ as the chord corresponding to the arc $\{x(\tau), 0 \leqslant \tau \leqslant t\}$ joining $x_- = x(0)$ and $x_+ = x(t)$. Of course, X is the reflection centre for this pair of phase space points [12].

Canonical transformations in phase space are generated implicitly by the centre action, i.e., the generating function $S(X)$ [12], such that

$$J\xi = \frac{\partial S}{\partial X}, \tag{A.3}$$

where

$$J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{A.4}$$

or by the chord action $S(\xi)$, such that

$$-JX = \frac{\partial S}{\partial \xi}. \tag{A.5}$$

Comparing with more familiar generating functions, e.g., $S(q_-, q_+)$,

$$\frac{\partial S}{\partial q_+} = p_+, \qquad \frac{\partial S}{\partial q_-} = p_-, \tag{A.6}$$

it follows that the conjugate variable to X should really be $J\xi$ rather than the chord itself. However, this lacks a clear geometrical interpretation in terms of $x_\pm$, so it is better to keep ξ, but replace all scalar products by skew products. Hence, in all Fourier transforms we use

$$(J\xi) \cdot X = \xi \wedge X = \xi_p \cdot Q - \xi_q \cdot P = \sum_{\ell=1}^{L} \left(\xi_{p_\ell} Q_\ell - \xi_{q_\ell} P_\ell\right). \tag{A.7}$$

Appendix B. Parity eigenstates

Any state $|\psi\rangle$ in Hilbert space can be decomposed into components of even or odd parity (eigenvalue +1, or -1) of any of the reflection operators $\hat{R}_X$. Indeed these projectors were presented by Royer [11] as

$$\hat{P}_\pm^X = \tfrac{1}{2}(1 \pm \pi\hbar \hat{R}_X), \tag{B.1}$$

so that the parity decomposition of an arbitrary density operator, $\hat{\rho}$, is

$$\hat{\rho}_\pm^X = \frac{\hat{P}_\pm^X \hat{\rho} \hat{P}_\pm^X}{\mathrm{tr}\, \hat{\rho} \hat{P}_\pm^X}. \tag{B.2}$$

Since these reduced density operators commute with the reflection operator, the Wigner function corresponding to $\hat{R}_X \hat{\rho}$ is

$$W_X^\pm(X') = 2^L \mathrm{tr}\, \hat{R}_{X'} \hat{R}_X \hat{\rho}_\pm^X = \pm 2^L \mathrm{tr}\, \hat{R}_{X'} \hat{\rho}_\pm^X = 2^L\, \mathrm{e}^{2\mathrm{i}X' \wedge X/\hbar} \mathrm{tr}\, \hat{T}_{2(X'-X)} \hat{\rho}_\pm^X, \tag{B.3}$$

where we used the general group relations between translations and reflections [12]. Thus, shifting the origin to the centre of symmetry, we obtain

$$W_0^\pm(X) = \pm 2^L \chi_0^\pm(-2X). \tag{B.4}$$

In the case that $\hat{\rho}_\pm^X$ is the projection of an arbitrary $\hat{\rho}$ according to (B.2), then the general form of the projected Wigner function given by [26] is transported to the chord function as

$$\chi_0^\pm(\xi) = \frac{\chi(\xi) + \chi(-\xi) \pm 2W(\xi/2)}{4\,[1 \pm \pi\hbar W(0)]}. \tag{B.5}$$

Evidently, all chord functions with pure parity are real. The reciprocal is also true, because the imaginary part of $\chi(\xi)$ cancels if and only if

$$\mathrm{tr}\,(\hat{T}_{-\xi} \hat{\rho}) = \chi(\xi) = \chi(\xi)^* = \chi(-\xi) = \mathrm{tr}\, \hat{T}_\xi \hat{\rho}. \tag{B.6}$$

But, using the group properties of translations and reflections,

$$\mathrm{tr}\, \hat{T}_{-\xi} \hat{\rho} = \mathrm{tr}\, \hat{R}_0 \hat{T}_\xi \hat{R}_0 \hat{\rho} = \mathrm{tr}\, \hat{T}_\xi \hat{R}_0 \hat{\rho} \hat{R}_0, \tag{B.7}$$

which is only equal to (B.6) for all ξ if $[\hat{\rho}, \hat{R}_0] = 0$.

A mixture of pure states, each of which has definite parity, commutes with $\hat{R}_0$ and hence produces a real chord function. The imaginary part of the chord function is related to the off-diagonal parity representation of the density matrix, i.e., for an orthogonal basis of odd and even states, this is the block of the density matrix coupling the different parities.

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